

On a Divisibility Property of Central Binomial Coefficients

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May 2026

Abstract

We study the divisibility properties of the expression $\frac{\binom{2q}{q}-2}{q}$ for integers $q > 2$. Computational evidence suggests that this quantity is an integer if and only if q is either a prime or the square of a prime. We provide partial results and structural analysis supporting this conjecture. The central binomial coefficients $\binom{2q}{q}$ play a central role in combinatorial and number theory. In this paper, we investigate the integrality of the proposed expression and characterize the integers q for which it holds.

1 Introduction

The central binomial coefficients are defined as:

$$\binom{2q}{q} = 3 \times 2^{q-1} \prod_{i=3}^q \left(2 - \frac{1}{i}\right) \wedge q > 2 \quad (1)$$

Their arithmetic properties have been extensively studied, particularly in relation to congruences. We focus on the integrality of $\frac{\binom{2q}{q}-2}{q}$.

2 Experimental Evidence

Numerical computations for $q \leq 10^5$ suggest the following pattern:

- If $q = p$ (prime), then $\frac{\binom{2q}{q}-2}{q} \in \mathbb{N}$.
- If $q = p^2$, the same behavior is observed.
- Otherwise, the expression is generally not an integer.

Examples:

- $q = 3$: $\frac{20-2}{3} = 6$ (Integer).
- $q = 5$: $\frac{252-2}{5} = 50$ (Integer).
- $q = 9 = 3^2$: Integer.
- $q = 15$: Not an integer.

3 Limitations of the Method and Structural Gaps

The approach relies on congruence properties under prime power moduli.

3.1 Dependence on Wolstenholme Type Congruences

A key ingredient for $q = p^2$ is the classical congruence $\sum_{i=1}^{p-1} \frac{1}{i} \equiv 0 \pmod{p^2}$ for $p \geq 5$. This ensures the cancellation of first-order terms, leading to $\binom{2p^2}{p^2} \equiv 2 \pmod{p^2}$. However, no analogous general congruence holds modulo p^3 .

3.2 Breakdown for Higher Prime Powers

For $q = p^i$ with $i \geq 3$, terms of order p^2, p^3 accumulate without sufficient cancellation in the product expansion. This leads to the failure of $\binom{2p^i}{p^i} \equiv 2 \pmod{p^i}$ for $i \geq 3$.

3.3 Structural Obstruction for Composite Integers

When $q = p_1^{a_1} \dots p_k^{a_k}$ ($k \geq 2$), the validity of $\binom{2q}{q} \equiv 2 \pmod{q}$ requires simultaneous congruences across all prime components. This is obstructed by base- p representation complexity and Kummer's theorem, as the number of carries varies across primes.

4 Main Conjecture

Conjecture 1. *Let $q > 2$. Then $\frac{\binom{2q}{q}-2}{q} \in \mathbb{N}$ if and only if $q = p^i$ for $i \in \{1, 2\}$, where p is a prime.*

5 Prime and Prime Square Cases

Proposition 1. *If $q = p$ is prime, then $\binom{2p}{p} \equiv 2 \pmod{p}$, following from Lucas' Theorem.*

Proposition 2. *If $q = p^2$, then $\binom{2p^2}{p^2} \equiv 2 \pmod{p^2}$. Using the product formula:*

$$\binom{2p^2}{p^2} = 2 \prod_{i=1}^{p^2-1} \left(1 + \frac{p^2}{i}\right) \equiv 2 \prod_{j=1}^{p-1} \left(1 + \frac{p}{j}\right) \pmod{p^2}. \quad (2)$$

By Wolstenholme's Theorem, $\sum_{j=1}^{p-1} \frac{1}{j} \equiv 0 \pmod{p^2}$, satisfying the condition.

6 Composite Integers and Failure

By Kummer's Theorem, the p -adic valuation $v_p\left(\binom{2q}{q}\right)$ equals the number of carries when summing $q + q$ in base p . For composite numbers, residues in the p -adic expansion prevent the uniform cancellation necessary for the $2 \pmod{q}$ congruence.

Example: For $q = 6$ (2×3), $\binom{12}{6} = 924$. Since $924 \equiv 0 \pmod{6}$ and $924 \not\equiv 2 \pmod{6}$, the expression is not an integer.

7 Conclusion

The Fragoso Conjecture is supported by the exceptional arithmetic structure of primes and prime squares. While it holds for p and p^2 , it fails for p^3 and composite integers due to the breakdown of harmonic sum cancellations and p-adic valuation inconsistencies.

References

- [1] GRAHAM, R. L.; KNUTH, D. E.; PATASHNIK, O. *Concrete Mathematics: A Foundation for Computer Science*. 2nd ed. Reading, MA: Addison-Wesley, 1994.